Satellite Lifetimes under the Influence of Continuous Thrust, Atmospheric Drag, and Planet Oblateness

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Approximate analytic solutions are obtained for the lifetimes of satellites in either circular or elliptical orbits under the influence of continuous tangential thrust, atmospheric drag, and planet oblateness. The cases treated are those in which the satellite mission requires the perigee of the orbit to remain within fixed bounds in altitude and position. For circular orbits it is found that, to increase greatly the useful lifetime over the nonthrusting case, an almost perfect matching of thrust to drag is required. Two solutions are obtained for elliptical orbits. The first requires that the thrust level over each revolution be the value needed to hold the perigee fixed. Extremely low thrust levels are found over most of the lifetime while increases in lifetime on the order of 30% over the nonthrusting case result. The second case considers thrusting at a constant level over a symmetrical arc about perigee; the length of the arc for each cycle being determined by the condition that the perigee remain fixed. Greatly increased lifetimes over the nonthrusting case and the preceding case are obtained with low mass consumption and small thrust-to-drag ratios. Expressions for the forces required to provide a secular change of the line of nodes, the line of apsides, and the inclination including the effect of planet oblateness are given and several examples are calculated.

Nomenclature

A	= effective drag area
a	= semimajor axis of orbit
c	= density scale factor
C_D	= drag coefficient
e	= orbit eccentricity
F_D	= tangential drag force
F_T	= total tangential force perturbation
F_t	= tangential thrust perturbation
$F_\perp$	= total force perturbation perpendicular to orbit
f	= F_t/F_D
δh	= bound on motion allowed perigee of elliptic orbit or radius of circular orbit
i	= orbit inclination to equator
I	= specific impulse
J	= coefficient of the second-order zonal harmonic in the earth's gravitational potential
m	= vehicle mass
dm/dN	= m', mass flow per revolution
N	= revolution number
N	= useful lifetime with thrust
N_0	= useful lifetime without thrust
p	= $a(1 - e^2)$
r	= radial distance from earth center
r_a	= apogee distance of orbit
r_p	= perigee distance of orbit
R_e	= earth's radius
t	= time
v	= vehicle velocity
z^2	= $(r_a - r_p)/c = (r_a + r_p)e/c$
β	= true anomaly on either side of perigee over which thrusting occurs
θ	= angle between the ascending node and the vehicle position
μ	= gravitational parameter
ρ	= atmospheric density
Ω	= longitude of the ascending node
ω	= angle between the line of nodes and the line of apsides

Subscripts

0	= in initial orbit
f	= in final orbit (end of useful lifetime)

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Introduction

VARIOUS satellite missions considered require extended lifetimes at perigee altitudes that are low and remain within fixed bounds. In addition, close control of the satellite orbit with regard to rotation of the line of nodes and the line of apsides will be needed.

To date, many authors[1–4] have treated the effects on satellites of drag and planet oblateness. It is the purpose of this paper to present a solution to the problem of controlling by continuous thrust the motion of a satellite perturbed by atmospheric drag and planet oblateness. The subject appears to have been first treated by Ehricke.[5]

The first section treats the force requirements for positioning of the line of nodes, the line of apsides, and the orbit inclination. As only secular changes in the orbit are considered in the paper, oblateness effects that are included here are not included in the remaining sections.

For a circular orbit and continuous tangential thrust, the problem is quite simple. The motion of the line of apsides is not a factor, and, if sufficient thrust is available to counteract the drag, the satellite will stay in orbit as long as propellant remains.†

If, however, there is an unbalance of drag over thrust caused by engine limitations or faulty control, it is necessary to know its effect on the satellite lifetime. The solution to this problem is given in the second section.

For elliptical orbits the problem is more complicated. The position of the line of apsides must be considered now along with control of the perigee distance. Moreover, although it is desirable to extend the lifetime as long as possible, it cannot be done by indiscriminately applying the largest thrust available over the entire cycle. This results because of the condition that the perigee remain within fixed bounds. As the drag is peaked about perigee, a very small force applied over the entire cycle is necessary to hold the perigee fixed, whereas the force required to hold the apogee fixed is much larger. Therefore, a force far too small to hold the apogee point still may cause the perigee distance to increase beyond the permitted limits resulting in a circularization process of a type not encountered during nonthrusting satellite motion.

† Since low altitudes and long lifetimes are being considered, the fraction of the useful satellite life remaining once thrust has terminated is very small.

With this in mind, the analysis treats two different thrusting programs for elliptic orbits in each of which the perigee distance is maintained constant. Variations of this condition of fixed perigee are of course possible, e.g., perigee oscillating within its bounds, but it is felt that if the bounds on the perigee are not large the main features of the trajectory will be illustrated by the thrust programs chosen. The first program considered treats a constant tangential thrust level about each cycle, the magnitude of the thrust being determined by the condition that the perigee remain fixed. The second thrust program provides for tangential thrusting along a symmetrical arc about perigee at a constant level for the entire trajectory, the length of the arc for each cycle being determined by the condition of fixed perigee. Since the latter program provides a more efficient means of slowing the inward movement of the apogee than the former, longer lifetimes can be expected. For both of these cases, the portion of the cycle over which thrusting in the plane of the orbit is not required is free to provide orbit rotation. The analogous problems for the case of impulsive thrust have been treated by the author in Ref. 6.

Procedure

The procedure used in the following analysis is to derive expressions for the secular rate of change per revolution of the quantities to be investigated. These equations are derived by integrating over one cycle of the motion the usual relations for their time derivatives obtained through variation of parameters.[7]

The pertinent equations expressed in terms of tangential forces F_T and forces orthogonal to the orbit plane $F_\perp$ are

$$\frac{d\Omega}{dt} = \frac{r}{(\mu p)^{1/2}} \frac{F_\perp}{m} \frac{\sin(\theta + \omega)}{\sin i} \tag{1a}$$

$$\frac{di}{dt} = \frac{r}{(\mu p)^{1/2}} \frac{F_\perp}{m} \cos(\theta + \omega) \tag{1b}$$

$$\frac{d\omega}{dt} = \frac{2}{ev} \frac{F_T}{m} \sin(\theta - \omega) - \frac{r}{(\mu p)^{1/2}} \frac{F_\perp}{m} \frac{\sin(\theta + \omega)}{\tan i} \tag{1c}$$

$$\frac{da}{dt} = \frac{2a^2 v}{\mu} \frac{F_T}{m} \tag{1d}$$

$$\frac{dr_p}{dt} = \frac{2r_p}{ev} \left(\frac{r - r_p}{r}\right) \frac{F_T}{m} \tag{1e}$$

$$\frac{dr_a}{dt} = \frac{2r_a}{ev} \left(\frac{r_a - r}{r}\right) \frac{F_T}{m} \tag{1f}$$

The normal force component has been set equal to zero as only tangential forces acting in the plane of the orbit are considered. In integrating these equations over one revolution, the following approximation has been adopted

$$dt = (rd\theta/v)[1 + 0(e^2)] \simeq rd\theta/v \tag{2}$$

since terms of order e^2 and higher are not retained in the analysis.

The atmospheric density is approximated analytically by

$$\rho(r) = \rho(R)e^{-[(r-R)c]} \tag{3}$$

and the values of c and R used obtained from the Air Research and Development Command 1959 Model Atmosphere so as to fit the altitude region of interest.

The drag force on the satellite is taken to be of the form

$$F_D = (C_D/2)\rho A v^2 \tag{4}$$

acting in a direction tangential but opposing the motion.

The mass of the satellite is assumed to decrease at a constant linear rate with the number of revolutions.

$$m = m_0 - (dm/dN)N = m_0 - m'N \tag{5}$$

Actually, the mass could be chosen as any function of N such that $dN/m(N)$ is integrable. The choice made does not represent a limitation of the analysis.

Using these relations expressions for the useful lifetimes of satellites in circular or elliptical orbits under continuous tangential thrust are now derived.

Section I. Orbit Correction and Oblateness Effects

The present section is concerned with the force requirements for positioning of the line of nodes and the line of apsides under continuous thrust. The secular changes in these quantities per revolution caused by the earth's oblateness are well known and given by

$$\begin{aligned} d\Omega/dN &= -2\pi(J/p^2)R_e^2 \cos i \\ d\omega/dN &= 2\pi(J/p^2)R_e^2(2 - \tfrac{5}{2}\sin^2 i) \end{aligned} \tag{6}$$

To obtain the effect of thrust on the orbit, Eqs. (1a–c) are integrated over one cycle assuming that the forces vary such that

$$\begin{aligned}
F_t &= F_{t_1} && 0 \le \theta - \omega < \pi \\
F_t &= F_{t_2} && \pi \le \theta - \omega < 2\pi \\
F_\perp &= F_{\perp_1} && 0 \le \theta + \omega < \pi \\
F_\perp &= F_{\perp_2} && \pi \le \theta + \omega < 2\pi \\
F_\perp &= \bar{F}_{\perp_1} && -\pi/2 \le \theta + \omega < \pi/2 \\
F_\perp &= \bar{F}_{\perp_2} && \pi/2 \le \theta + \omega < 3\pi/2
\end{aligned} \tag{7}$$

Adding the resulting equations to Eqs. (6), we have for the total secular change per revolution through $0(e)$

$$\frac{d\Omega}{dN} = -2\pi\frac{J}{a^2} R_e^2 \cos i + \frac{2a^2}{\mu m \sin i}(F_{\perp_1} - F_{\perp_2}) - \frac{3\pi e a^2}{2\mu m \sin i}(F_{\perp_1} + F_{\perp_2} + \bar{F}_{\perp_1} + \bar{F}_{\perp_2}) \sin 2\omega \tag{8a}$$

$$\frac{d\omega}{dN} = 2\pi\frac{J}{a^2} R_e^2\left(2 - \frac{5}{2}\sin^2 i\right) + \frac{4a^2}{e\mu m}(F_{t_1} - F_{t_2}) - \frac{2a^2}{\mu m \tan i}(F_{\perp_1} - F_{\perp_2}) + \frac{3\pi e a^2}{2\mu m \tan i} \times (F_{\perp_1} + F_{\perp_2} + \bar{F}_{\perp_1} + \bar{F}_{\perp_2}) \sin 2\omega \tag{8b}$$

$$\frac{di}{dN} = \frac{2a^2}{\mu m}(\bar{F}_{\perp_1} - \bar{F}_{\perp_2}) - \frac{3\pi e a^2}{2\mu m}(F_{\perp_1} + F_{\perp_2} + \bar{F}_{\perp_1} + \bar{F}_{\perp_2}) \tag{8c}$$

Equation (8b) does not include a drag term since it can be expected that the drag will be almost symmetrical about perigee and, hence, have little effect on the position of the line of apsides.

To determine the magnitude of the quantities involved, results pertaining to a polar orbit and an orbit with inclination to the equator of 70° are presented. Numerical values obtained use a perigee altitude of 110 miles and, for the rotation of the line of apsides, an eccentricity of 0.1. A vehicle weight of 3900 lb is used for the calculation of thrust levels.

Using Eq. (8a) it is found that for the polar orbit the earth's oblateness causes no rotation of the line of nodes whereas for the orbit of 70° inclination there is a regression of 3.1° per day. Setting $F_{\perp_2} = -F_{\perp_1}$, with $\bar{F}_{\perp_1} = \bar{F}_{\perp_2} = 0$, it is found that a force of magnitude $F_{\perp_1} = 2.9$ lb is required to stop the rotation.

Consideration of the motion of the line of apsides shows that the earth's oblateness causes the apsides of a polar orbit of eccentricity, 0.1, to regress 3.9° per day while for the orbit of 70° inclination a regression of 1.6° per day is found. Taking $F_{\perp_1} = F_{\perp_2} = \bar{F}_{\perp_1} = \bar{F}_{\perp_2} = 0$ it is determined that a force differential of $F_{t_1} - F_{t_2} = 0.31$ lb will restrain the polar orbit while the orbit of 70° inclination requires $F_{t_1} - F_{t_2} = 0.13$ lb. In addition, to obtain an inclination change of 1° per day requires $\bar{F}_{\perp_1} = 0.99$ lb with $\bar{F}_{\perp_2} = -\bar{F}_{\perp_1}$ and $F_{\perp_1} = F_{\perp_2} = 0$.

Section II. Circular Orbits

Integrating Eq. (1d) over 1 cycle and setting $a = r$, one obtains

$$dr/dN = (4\pi r^3/\mu m)F_T \tag{9}$$

The total tangential force is made of the thrust force F_t and the drag force F_D

$$F_T = F_t - (C_D/2)\rho A v^2 \tag{10}$$

Therefore

$$\frac{dr}{dN} = \frac{4\pi r^3}{\mu m} F_t \left\{1 - \frac{C_D}{2} \frac{A\rho(R)e^{-[(r-R)/c]}}{F_t}\right\} \tag{11}$$

The useful life is determined by the condition that the difference between the final and initial satellite radii be some fixed value δh.

$$r_0 - r_f = \delta h \tag{12}$$

It will be assumed that $\delta h/r \ll 1$ during the useful lifetime. Therefore, taking $r = r_0$ in Eq. (11) except in the exponential term one obtains upon integration

$$-\frac{4\pi r_0^3 F_t}{\mu c m'} \ln\left(1 - \frac{m'N}{m_0}\right) = \ln\left\{\frac{1 - fe^{-[(h_0-h)/c]}}{1 - f}\right\} \tag{13}$$

where

$$f = F_t/F_D(r_0) \qquad h = r - R$$

If $(m'N/m_0) \ll 1$ the left-hand side of Eq. (13) can be expanded yielding

$$\frac{C_D A}{m_0} N = \frac{c}{2\pi\rho(r_0)r_0^2 f} \ln\left\{\frac{1 - fe^{-[(h_0-h)/c]}}{1 - f}\right\} \tag{14}$$

The comparable expression with no thrust acting is obtained by taking the limit of Eq. (14) as $f \to 0$ giving

$$\frac{C_D A}{m_0} N = \frac{c}{2\pi\rho(r_0)r_0^2}\{1 - e^{-[(h_0-h)/c]}\} \tag{15}$$

Taking $h_0 - h = \delta h$, the number of revolutions of useful life with thrust $\bar{N}$ and the useful life without thrust N_0 are determined by Eqs. (14,15). Results are given in Fig. (1) where the ratio of the lifetimes with and without thrust $\bar{N}/N_0$ is plotted vs the thrust-to-drag ratio used for various initial altitudes assuming $m'\bar{N}/m_0 \ll 1$.

Once the ratio of thrust to initial drag to be used has been chosen, the amount of propellant necessary is determined easily. The thrust is given by

$$F_t = -Ig_{00}\dot{m} \tag{16}$$

Integrating $\dot{m}$ over one cycle, it is found that

$$dm/dN = -(2\pi r^{3/2}/\mu^{1/2}Ig_{00})F_t \tag{17}$$

from which, since the variation in r has been assumed small, one has

$$m_0 - m_f = (2\pi\bar{r}^{3/2}/\mu^{1/2}Ig_{00})F_t N \tag{18}$$

where $\bar{r}$ is the average value of r during the N revolutions.

This result could be viewed as obtained by considering

$$I = I_0 r^{3/2}/\bar{r}^{3/2} \tag{19}$$

in which case dm/dN would be exactly a constant.

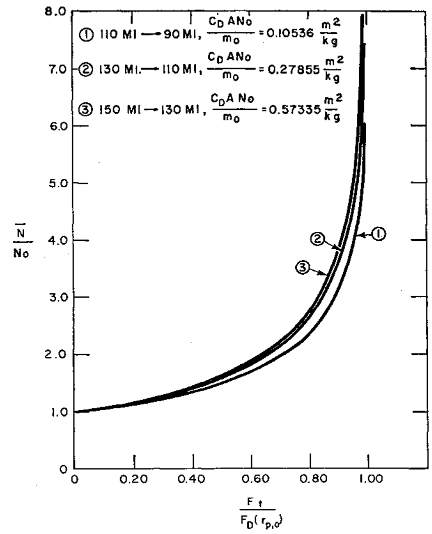

Fig. 1 The ratio of the number of revolutions of useful life with thrust to that without thrust vs thrust to initial drag ratio used for circular orbits assuming $m'\bar{N}/m_0 \ll 1$

Section III. Elliptic Orbits

In working with elliptic orbits the dependent variables used are the apogee and perigee distances r_a and r_p. Equations (1e,f) are integrated over 1 cycle retaining terms through $0(e)$ assuming the drag to be strongly peaked about perigee. Under this assumption r is expanded about perigee for use in the exponential of the density function obtaining

$$r \simeq r_p\{1 + [e/(1+e)][(\theta - \omega)^2/2]\} \tag{20}$$

Two thrust programs are considered both of which maintain a fixed perigee distance. The first program treated considers the force constant over each cycle, its value to be determined by the condition that the perigee remain fixed.

Defining $z^2 = (r_a - r_p)/c$, the integration over the cycle of Eqs. (1e, f) yields

$$\frac{dr_p}{dN} = \frac{4\pi F_t r_a r_p^2}{\mu m}\left(1 + \frac{c}{2(r_a + r_p)} z^2\right) - \frac{(\pi)^{1/2} C_D A\rho(r_p) r_p(r_a + r_p)}{m}\left[\frac{r_a}{r_p}\right]^{1/2}\frac{1}{z^3} \tag{21a}$$

$$\frac{dr_a}{dN} = 4\pi F_t r_p r_a^2\left(1 - \frac{c}{2(r_a + r_p)} z^2\right) - \frac{2(\pi)^{1/2} C_D A\rho(r_p) r_a(r_a + r_p)}{m}\left[\frac{r_a}{r_p}\right]^{1/2}\frac{1}{z} \tag{21b}$$

where, because of the peaking of the drag about perigee, the limits of $-\pi \leq \theta - \omega \leq \pi$ are approximated by $-\infty \leq \theta - \omega \leq \infty$.

These equations, exclusive of the terms containing the thrust, have been derived by El'yasberg[8] in a similar manner and shown to be accurate descriptions of the motion.

Setting $dr_p/dN = 0$, solving Eq. (21a) for F_t, and substituting into Eq. (21b), one obtains

$$\frac{dz^2}{dN} = \frac{2C_D A\alpha}{mz}\left\{\frac{1}{z^2} - \left(2 + \frac{c}{r_a + r_p}\right)\right\} \tag{22}$$

where

$$\alpha = \frac{(\pi)^{1/2}}{2}\frac{\rho(r_p)\bar{r}_a(\bar{r}_a + r_p)}{c}\left[\frac{\bar{r}_a}{r_p}\right]^{1/2}$$

or

$$\frac{C_D A}{m}\alpha dN = \frac{z^4 dz}{1 - \{2 + [c/(\bar{r}_a + r_p)]\}z^2} \tag{23}$$

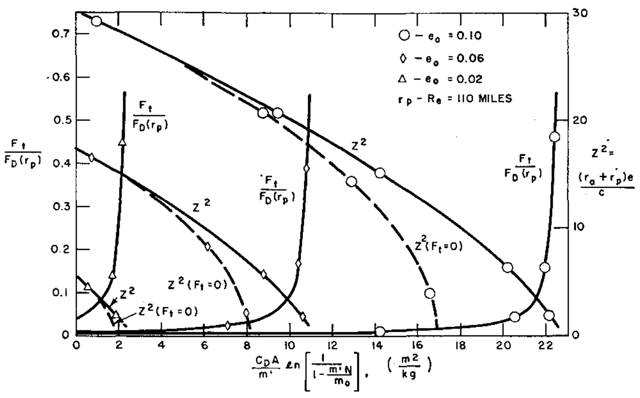

Fig. 2 For fixed perigee and continuous thrusting over entire cycle thrust to drag ratio and orbit eccentricity vs revolutions factor for several initial eccentricities

To integrate this equation, r_a is replaced by its average value over the lifetime, and only variations of the new dependent variable z are considered.

Letting

$$b = 2 + [c/(\bar{r}_a + r_p)]$$

one obtains upon integration

$$-\frac{C_DA}{m'}\,b\alpha\ln\left(1-\frac{m'N}{m_0}\right) \equiv J(z) = \frac{1}{3}(z_0{}^3 - z^3) + \frac{1}{b}(z_0 - z) + \frac{1}{b^{5/2}}\ln\left\{\left[\frac{(b)^{1/2}z_0 - 1}{(b)^{1/2}z_0 + 1}\right]\left[\frac{(b)^{1/2}z + 1}{(b)^{1/2}z - 1}\right]\right\} \quad (24)$$

where if $m'N/m_0 \ll 1$ it is found that

$$(C_DA/m_0)N = (1/\alpha b)J(z) \quad (25)$$

From Eq. (21a) with $dr_p/dN = 0$ one has

$$\frac{F_t}{F_D(r_p)} = \frac{F_t}{(C_D/2)\rho(r_p)Av^2(r_p)} = \frac{1}{(\pi)^{1/2}}\left(\frac{\bar{r}_a + r_p}{\bar{r}_a}\right)\left[\frac{\bar{r}_a}{r_p}\right]^{1/2}\frac{1}{z^3\{2 + [cz^2/(\bar{r}_a + r_p)]\}} \quad (26)$$

where $v(r_p)$ is the circular velocity at perigee radius.

The lifetime will be considered over when $z = z_f = 1$ as this corresponds to an eccentricity no larger than 4×10^{-3}, an approximately circular orbit. If propellant still remains, the lifetime corresponding to the spiral phase at a given thrust level may be added to the value determined previously.

Curves of $F_t/F_D(r_p)$ and z^2 vs

$$(C_DA/m')\ln\{1/[1 - (m'N/m_0)]\}$$

for the forementioned solution are given in Fig. (2) for initial eccentricities of $e_0 = 0.02$, 0.06, and 0.1 and with a perigee altitude of 110 miles. The curves of z^2 vs

$$(C_DA/m')\ln[1/1 - (m'N/m_0)]$$

assuming $m'N/m_0 \ll 1$ are compared with those obtained from the analysis of Ref. 7 for the nonthrusting case where the perigee is allowed to collapse to 20 miles below its original position of 110-mile alt. By thrusting in the forementioned manner, approximately a 30% increase in lifetime over the nonthrusting case in each of the three examples taken was found.

It should be noted that the forementioned thrusting program is relatively inefficient in holding the apogee out. This is to be expected since little of the total energy input by thrust per cycle is applied around perigee where it would have the greatest affect on maintaining the apogee distance. Further, as Fig. 2 shows, very low thrust levels must be maintained for each cycle over most of the lifetime or the perigee will start moving out.

To overcome these difficulties, a different thrust program is suggested. Thrusting will proceed now at a constant level for the entire orbit along a symmetrical arc about perigee, i.e., within a true anomaly of β on either side of perigee; the value of β for each cycle is to be determined by the condition that the perigee remain fixed. The remainder of the orbit from β to $2\pi - \beta$ is available for corrections perpendicular to the plane of the orbit. The lifetime of the trajectory is considered to be the number of revolutions obtained between the initial $\beta = \beta_0$ and $\beta = \pi$ where thrusting occurs over the entire cycle.

Integration of Eqs. (1e,f) from $-\beta \leq \theta - \omega \leq \beta$ for the thrust term and from $-\infty \leq \theta - \omega \leq \infty$ for the drag term retaining quantities through $0(e)$ yields

$$\frac{dr_p}{dN} = \frac{4r_ar_p{}^2F_t}{\mu m}\left\{\beta\left(1 + \frac{e}{2}\right) - (1 + 2e)\sin\beta + \frac{3}{4}e\sin2\beta\right\} - \frac{(\pi)^{1/2}C_DA\rho(r_p)r_p(r_a + r_p)}{m}\left[\frac{r_a}{r_p}\right]^{1/2}\frac{1}{z^3} \quad (27a)$$

$$\frac{dr_a}{dN} = \frac{4r_pr_a{}^2F_t}{\mu m}\left\{\beta\left(1 - \frac{e}{2}\right) + (1 - 2e)\sin\beta - \frac{3}{4}e\sin2\beta\right\} - \frac{2(\pi)^{1/2}C_DA\rho(r_p)r_a(r_a + r_p)}{m}\left[\frac{r_a}{r_p}\right]^{1/2}\frac{1}{z} \quad (27b)$$

Setting

$$g(\beta) = \beta[1 + (e/2)] - (1 + 2e)\sin\beta + \tfrac{3}{4}e\sin2\beta \quad (28)$$

the function is represented approximately by

$$g(\beta) \simeq \beta^3g_1(\beta_0) \quad (29)$$

where

$$g_1(\beta_0) = \frac{1}{\beta_0{}^3}\{\beta_0[1 + (e_0/2)] - (1 + 2e_0)\sin\beta_0 + \tfrac{3}{4}e_0\sin2\beta_0\} \quad (30)$$

Putting $dr_p/dN = 0$ in Eq. (27a) and solving for β one obtains approximately

$$\beta = \gamma/z \quad (31)$$

where

$$\gamma^3 = \frac{(\pi)^{1/2}}{2fg_1(\beta_0)}\left(\frac{r_a + r_p}{r_a}\right)\left[\frac{r_a}{r_p}\right]^{1/2}$$

$$f = \frac{F_t}{F_D(r_p)} = \frac{F_t}{(C_D/2)\rho(r_p)Av^2(r_p)}$$

The effect on the solution of the forementioned approximation for $g(\beta)$ can be seen from Fig. 3 where (β approx/β true) vs β approx is plotted assuming an initial β of $\pi/4$ and an initial eccentricity of 0.1 that decreases inversely with β^2. It is seen that the approximation used for $g(\beta)$ yields a value of β within 10% of the true value for a given z.

Multiplying Eq. (27a) by r_a, Eq. (27b) by r_p, and adding with $dr_p/dN = 0$, one obtains

$$cr_p\frac{dz^2}{dN} = \frac{4r_a{}^2r_p{}^2F_t}{\mu m}\left[2\beta\left(1 - 2\frac{e\sin\beta}{\beta}\right)\right] - \frac{(\pi)^{1/2}C_DA\rho(r_p)r_ar_p(r_a + r_p)}{m}\left[\frac{r_a}{r_p}\right]^{1/2}\left[\frac{2}{z^3} + \frac{1}{z}\right] \quad (32)$$

Letting

$$h(\beta) = 2\beta[1 - (2e\sin\beta/\beta)] \quad (33)$$

the function is approximated by

$$h(\beta) \simeq 2\beta\,h_1(\beta_0) \quad (34)$$

where

$$h_1(\beta_0) = 1 - (2e_0\sin\beta_0/\beta_0)$$

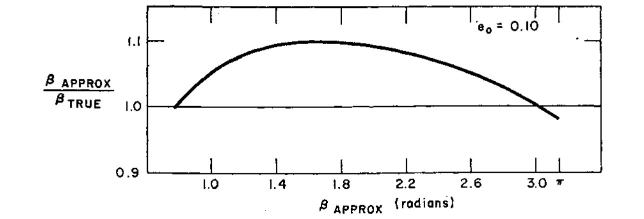

Fig. 3 Error entailed by the use of the approximate function for $g(\beta)$

Since the approximate value of $h(\beta)$ is always less than the true value for $\beta > \beta_0$, the approximation made again produces conservative results. Separating variables, Eq. (32) may be written as

$$-z^4dz/[(2C_2 - C_1)z^2 + C_2] = (C_DA/m)dN \tag{35}$$

where

$$C_1 = \frac{2\bar{r}_a{}^2 f\rho(r_p)h_1(\beta_0)\gamma}{c}$$

$$C_2 = \frac{(\gamma\pi)^{1/2}}{2}\frac{\rho(r_p)\bar{r}_a(\bar{r}_a + r_p)}{c}\left[\frac{r_a}{r_p}\right]^{1/2}$$

$$C_3 = 2C_2 - C_1$$

Integrating Eq. (35) between z_0 and z_f where z_0 and z_f are determined from Eq. (31) with β equal to β_0 and β_f, respectively, one obtains

$$\frac{C_DA}{m'}\ln\left[\frac{1}{1-(m'N_f/m_0)}\right] = \frac{1}{3C_3}(z_0{}^3 - z_f{}^3) - \frac{C_2}{C_3{}^2}(z_0 - z_f) + \frac{C_2{}^2}{C_3{}^2}\left(\frac{C_3}{C_2}\right)^{1/2} \times \left\{\tan^{-1}z_0\left(\frac{C_3}{C_2}\right)^{1/2} - \tan^{-1}z_f\left(\frac{C_3}{C_2}\right)^{1/2}\right\} \tag{36}$$

The quantity r_a occurring in C_1 and C_2 is approximated by its average value to perform the integration. In addition, the integration has assumed $C_3 > 0$. If f is increased until $C_3 = 0$, one finds

$$\frac{C_DA}{m'}\ln\left[\frac{1}{1-(m'N/m_0)}\right] = \frac{1}{5C_2}(z_0{}^5 - z_f{}^5) \tag{37}$$

A further increase in f such that $C_3 < 0$ causes the apogee to move out provided

$$z_0{}^2 > C_2/(C_1 - 2C_2) \tag{38}$$

Results are presented in Fig. 4 where f is plotted vs

$$(C_DA/m')\ln[1/1 - (m'N_f/m_0)]$$

for initial eccentricities of 0.02, 0.06, and 0.1 with a perigee altitude of 110 miles. It is seen that major increases in lifetime over the preceding thrust program and the nonthrusting case can be achieved.

These integrations were performed with m' taken as constant. This implies that as β varies I and $\dot{m}$ change so as to maintain m' and F_t constant. The propellant used may be calculated from

$$dm = -\dot{m}dt \simeq -(\dot{m}r/v)d\theta \tag{39}$$

which on integrating over one cycle from $-\beta \leq \theta - \omega \leq \beta$ yields

$$dm/dN = -(\dot{m}a^{3/2}/\mu^{1/2})[2\beta - 4e\sin\beta] \tag{40}$$

If dm/dN and F_t are constant, then

$$\dot{m} = \dot{m}_0\frac{(\beta_0 - 2e_0\sin\beta_0)}{(\beta - 2e\sin\beta)} \tag{41}$$

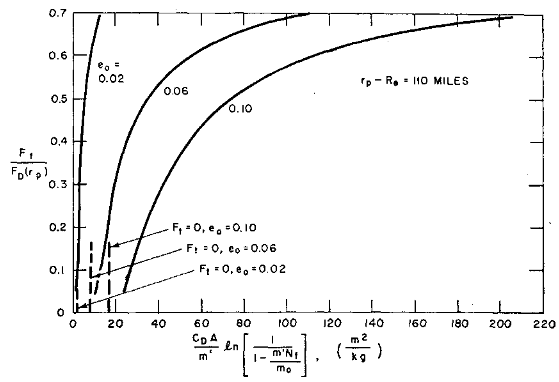

Fig. 4 For fixed perigee and continuous constant thrust over symmetrical arc about perigee thrust to drag ratio used vs revolutions factor for several initial eccentricities

and

$$I = \frac{F_t}{\dot{m}_0 g_{00}}\frac{(\beta - 2e\sin\beta)}{(\beta_0 - 2e_0\sin\beta_0)} \tag{42}$$

with which the expression for the mass ratio becomes

$$m_0/m_f = 1 + (\dot{m}_0 a^{3/2}/\mu^{1/2})[2\beta_0 - 4e_0\sin\beta_0]N_f \tag{43}$$

If the analysis is not restricted to constant dm/dN but rather a constant mass flow rate and a constant specific impulse I are taken to obtain a constant thrust, the mass ratio is calculated in the following manner.

Equation (40) is approximated by

$$dm/dN = -(\dot{m}a^{3/2}/\mu^{1/2})2\beta \tag{44}$$

which, by giving a value for dm/dN greater than the true one, provides a mass ratio larger than actually exists. The value obtained for m_0/m_f is therefore a safe estimate of the mass requirements for propulsion.

Substitution for β from Eq. (31) and using Eq. (34) to eliminate dN, Eq. (44) becomes

$$dm/m = [kz^3dz/(C_3z^2 + C_2)] \tag{45}$$

where C_2 and C_3 are as defined previously, $C_3 > 0$, and

$$k = \frac{2\dot{m}\bar{a}^{3/2}\gamma}{\mu^{1/2}A} = \frac{2\bar{a}^{3/2}\gamma\rho(r_p)\mu^{1/2}f}{Ig_{00}r_p} \tag{46}$$

Integrating between initial and final states one obtains

$$\ln\frac{m_0}{m_f} = \frac{k}{2C_3}\left\{(z_0{}^2 - z_f{}^2) - \frac{C_2}{C_3}\ln\left[\frac{1+(C_3/C_2)z_0{}^2}{1+(C_3/C_2)z_f{}^2}\right]\right\} \tag{47}$$

from which the required mass ratio may be determined. In Fig. 5 the ratio of thrust to drag force used is plotted vs final

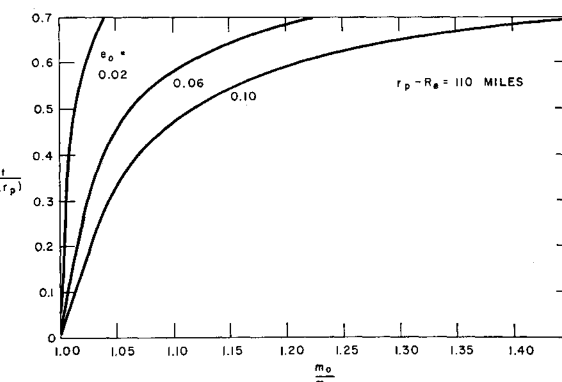

Fig. 5 For fixed perigee and continuous constant thrust over symmetrical arc about perigee thrust to drag ratio used vs final mass ratio for several initial eccentricities assuming constant mass flow rate

mass ratio for initial eccentricities of 0.02, 0.06, and 0.1 with a perigee altitude of 110 miles.

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Orbital Docking Dynamics

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A docking configuration and docking sequence are described for accomplishing an orbital refueling mission. The docking configuration consists essentially of matched conical surfaces, on the docking vehicles, and a latching device. A mathematical model for studying the collision dynamics is presented, and the effects of guidance errors, attitude errors, and various structural parameters on docking performance are discussed. Requirements for attitude control also are included. It is concluded that the proposed docking configuration is feasible and will accommodate reasonable system errors.

Introduction

THE first orbital docking missions will be performed in order to develop techniques and establish operational procedures as well as to accomplish logistic operations between orbiting vehicles. A typical docking mission will consist presumably of ground tracking to establish ephemerides of the target satellite, launch and ascent guidance of the chaser vehicle to an orbit "nearby" that of the target, terminal or rendezvous guidance to bring the chaser near the target, a docking maneuver to connect the vehicles, and finally a transfer of material from the chaser vehicle to the target. A subsequent disengagement of the vehicles also may be desirable. Of these various phases, all but docking and material transfer have been studied to some degree, and accounts of these studies appear in the literature. This probably can be attributed to the fact that these two phases simply occur later in the natural sequence of events leading to sustained orbital operations. Docking and material transfer techniques must be developed, however, before an integrated system for orbital operation can be designed.

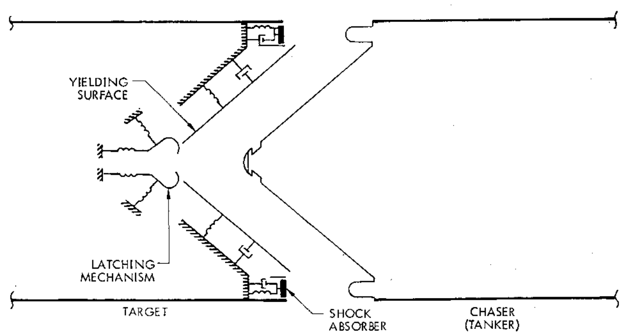

Fig. 1 Schematic cross section of docking assembly

This paper is concerned primarily with the dynamics of a docking collision between vehicles of particular specifications and the accompanying attitude control problem. To place these subjects in their proper context and perspective, the docking assembly is introduced first and the docking sequence summarized. The model used to study the docking collision then is described and the assumptions and parameters used in the collision study are presented. The results of this study and the attendant control requirements then are discussed.

Docking Description

The docking configuration described below represents a possible design for accomplishing a refueling operation. The configuration (Fig. 1) consists of matched conical docking surfaces on the target and chaser (tanker) vehicles, a shock absorber system around the rim of the target cone, and a latching mechanism located near the vertex of this cone. The docking surface of the target is attached by a compliant system of structural members enabling the surface to yield to the forces of collision. During the docking maneuver the docking surfaces will contact and move into near spatial coincidence, triggering the spring loaded latching mechanism into place. The latching mechanism is spring loaded both transversely and radially so that it not only secures the nose of the chaser vehicle but also tends to bring the vehicles into closer contact. The geometrical placement of the latching mechanism and the shock absorber is such that the chaser rim will contact the shock absorbers at almost the same time the latching device is triggered. The combined effect of the highly damped absorbers and the stabilizing axial force exerted by the latching mechanism will reduce the relative velocities of the vehicles. The latching mechanism finally is drawn into the target vehicle hydraulically and the vehicles securely fastened by additional connectors. Refueling is accomplished through propellant fittings located at the vertices of the docking cones.

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